Non-Commutative Scalar Fields

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1 Introduction

Finite Difference Methods for Scalar Fields in Non-Commutative Spaces: Numerical Computation of Mixed Derivatives and Action

`https://github.com/sphereofrealization/PythonCode/blob/main/Non_Commutative_Scaler_Fields.ipynb`

Abstract

In this paper, we explore numerical methods for simulating scalar field configurations in non-commutative two-dimensional spaces. We focus on the finite difference techniques employed to compute mixed partial derivatives and the action functional in the presence of non-commutative corrections. The methods presented address the challenges posed by non-commutative geometry, specifically in computing the mixed derivative terms that arise due to the deformation of spatial coordinates. We introduce semi-implicit time-stepping schemes to ensure numerical stability when dealing with stiff nonlinear terms. The approaches discussed here provide a framework for simulating and analyzing physical systems influenced by non-commutativity, which are not extensively documented in existing literature.

Introduction

Non-commutative geometry has attracted significant interest in theoretical physics, particularly in the context of field theories where spatial coordinates no longer commute. This deformation leads to modifications in the dynamics of scalar fields, introducing additional terms in the equations of motion that account for the non-commutative nature of space. The study of such systems requires novel numerical methods to accurately capture the effects of non-commutativity, especially when dealing with mixed derivative terms that are not present in commutative spaces.

In this paper, we present finite difference methods tailored for computing mixed partial derivatives in two-dimensional non-commutative spaces. We also discuss the numerical computation of the action functional over time, which is essential for analyzing the dynamical behavior of scalar fields under non-commutative corrections. Our focus is on the mathematical techniques employed in these computations, particularly the derivation and implementation of finite difference schemes for mixed derivatives and the integration of action in a discretized spatial domain.

Mathematical Formulation

Scalar Field Dynamics in Non-Commutative Space

Consider a real scalar field $\phi(x, y, t)$ defined over a two-dimensional non-commutative space. The non-commutativity is characterized by the relation $[x, y] = i\theta$, where θ is a constant parameter representing the deformation of space. In the context of scalar field theories, this non-commutativity introduces modifications to the equations of motion, resulting in additional terms involving mixed derivatives of the field.

The action functional S for such a scalar field with a quartic self-interaction and non-commutative correction can be written as:

$$S = \int dt \int dx\, dy \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - V(\phi) + \epsilon\theta(\partial_x \phi)(\partial_y \phi) \right),$$

where $\nabla\phi$ denotes the gradient of ϕ, $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4$ is the potential energy density, m is the mass parameter, λ is the self-interaction coupling, ϵ is the non-commutative correction strength, and θ is the non-commutative parameter.

The corresponding equation of motion derived from the Euler-Lagrange equation is:

$$\partial_t^2 \phi = -\left(\Delta\phi + m^2\phi + \frac{\lambda}{6}\phi^3 + \epsilon\theta\partial_x\partial_y\phi \right),$$

where Δ is the Laplacian operator.

Numerical Challenges

The presence of the mixed derivative term $\partial_x\partial_y\phi$ due to non-commutativity presents a challenge for numerical computation. Standard finite difference methods primarily focus on computing spatial derivatives independently in each dimension. Accurately approximating mixed derivatives requires careful consideration to maintain consistency and stability in the numerical scheme.

Additionally, the nonlinear nature of the self-interaction term ϕ^3 and the potential for stiffness in the equations necessitate the use of stable time-stepping methods. We employ semi-implicit schemes to address stability issues, particularly when simulating over extended periods.

Finite Difference Approximation of Mixed Derivatives

Standard Finite Difference Operators

For a scalar field $\phi(x, y)$ discretized on a uniform grid with spacing dx and dy in the x and y directions respectively, the standard finite difference approximations for the first-order partial derivatives are:

$$\partial_x \phi \approx \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2dx},$$
$$\partial_y \phi \approx \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2dy}.$$

The second-order partial derivatives (Laplacian) are approximated as:

$$\partial_x^2 \phi \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{dx^2},$$
$$\partial_y^2 \phi \approx \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{dy^2}.$$

Novel Finite Difference Scheme for Mixed Derivatives

The mixed partial derivative $\partial_x \partial_y \phi$ requires careful discretization to ensure accuracy and stability. The challenge lies in constructing a finite difference operator that approximates the mixed derivative using grid point values while minimizing truncation errors.

We propose a finite difference scheme that computes the mixed derivative by first approximating the first-order derivatives and then differentiating these approximations with respect to the other variable. The steps are as follows:

1. **Compute Intermediate First-Order Derivatives:**

The forward and backward differences for $\partial_x \phi$ and $\partial_y \phi$ are computed to enhance accuracy:

$$(\partial_x \phi)_{\text{forward}} \approx \frac{\phi_{i+1,j} - \phi_{i,j}}{dx},$$
$$(\partial_x \phi)_{\text{backward}} \approx \frac{\phi_{i,j} - \phi_{i-1,j}}{dx},$$
$$(\partial_x \phi)_{\text{average}} \approx \frac{(\partial_x \phi)_{\text{forward}} + (\partial_x \phi)_{\text{backward}}}{2}.$$

Similarly for $\partial_y \phi$:

$$(\partial_y \phi)_{\text{forward}} \approx \frac{\phi_{i,j+1} - \phi_{i,j}}{dy},$$
$$(\partial_y \phi)_{\text{backward}} \approx \frac{\phi_{i,j} - \phi_{i,j-1}}{dy},$$
$$(\partial_y \phi)_{\text{average}} \approx \frac{(\partial_y \phi)_{\text{forward}} + (\partial_y \phi)_{\text{backward}}}{2}.$$

2. **Compute Mixed Derivative:**

The mixed partial derivative is approximated by differentiating $(\partial_x \phi)_{\text{average}}$ with respect to y:

$$\partial_x \partial_y \phi \approx \frac{(\partial_x \phi)_{i,j+1} - (\partial_x \phi)_{i,j-1}}{2dy}.$$

This method ensures that the mixed derivative captures the change in the first-order derivative $\partial_x \phi$ along the y-direction, and vice versa.

Justification and Accuracy

This finite difference scheme for $\partial_x \partial_y \phi$ is derived from central difference approximations and ensures second-order accuracy in both dx and dy. By averaging the forward and backward differences, we reduce the truncation error associated with asymmetric difference approximations.

Let us analyze the truncation error of the mixed derivative approximation. For smooth functions $\phi(x, y)$, the Taylor series expansion yields:

$$\phi_{i+1,j} = \phi_{i,j} + dx\,(\partial_x\phi)_{i,j} + \frac{dx^2}{2}\left(\partial_x^2\phi\right)_{i,j} + O(dx^3),$$
$$\phi_{i-1,j} = \phi_{i,j} - dx\,(\partial_x\phi)_{i,j} + \frac{dx^2}{2}\left(\partial_x^2\phi\right)_{i,j} - O(dx^3).$$

Subtracting these expansions and dividing by $2dx$ gives the central difference approximation for $\partial_x\phi$ with an error of $O(dx^2)$. A similar analysis applies to $\partial_y\phi$.

By differentiating the central difference approximation of $\partial_x\phi$ with respect to y using central differences, we maintain second-order accuracy for the mixed derivative. Hence, the proposed scheme is consistent and accurate for smooth functions.

Semi-Implicit Time-Stepping Scheme

Stability Considerations

The equations of motion involve stiff nonlinear terms, particularly the self-interaction term ϕ^3 and the non-commutative correction involving the mixed derivative. Explicit time-stepping methods with large time steps can lead to numerical instability and divergence.

To enhance stability, we employ a semi-implicit time-stepping scheme that treats the linear terms implicitly and the nonlinear terms explicitly. We introduce an averaging of the field between the current and previous time steps to linearize the nonlinear terms partially.

Implementation of Semi-Implicit Scheme

Let ϕ^n denote the field at the current time step n, and ϕ^{n-1} at the previous step. The update equation for the field is:

$$\phi^{n+1} = \phi^n + \Delta t\left(-\left(\Delta\phi^n + m^2\phi^{n+1} + \frac{\lambda}{6}(\phi^{n+\frac{1}{2}})^3 + \epsilon\theta\partial_x\partial_y\phi^n\right)\right),$$

where $\phi^{n+\frac{1}{2}} = \frac{1}{2}(\phi^n + \phi^{n-1})$ is the average field. Rearranging terms, we solve for ϕ^{n+1}:

$$\phi^{n+1} = \frac{\phi^n - \Delta t\left(\Delta\phi^n + \frac{\lambda}{6}(\phi^{n+\frac{1}{2}})^3 + \epsilon\theta\partial_x\partial_y\phi^n\right)}{1 + \Delta t\, m^2}.$$

This implicit treatment of the linear mass term $m^2\phi^{n+1}$ enhances stability, allowing for larger time steps compared to fully explicit schemes. The nonlinear term is approximated using the averaged field to mitigate stiffness while keeping the computation tractable.

Numerical Computation of the Action Functional

Discretization of the Lagrangian Density

The action functional S is defined as the integral over spacetime of the Lagrangian density L:

$$S = \int dt \int dx\, dy\, L(x, y, t).$$

For numerical computation, we discretize this integral using finite difference approximations for derivatives and quadrature rules for integration over the spatial domain. The Lagrangian density at each grid point is computed as:

$$L_{i,j} = \frac{1}{2}(\partial_t \phi_{i,j})^2 - \frac{1}{2}\left((\partial_x \phi_{i,j})^2 + (\partial_y \phi_{i,j})^2\right) - V(\phi_{i,j}) + \epsilon\theta(\partial_x \phi_{i,j})(\partial_y \phi_{i,j}),$$

where $V(\phi_{i,j})$ is the potential energy density at grid point (i, j).

Numerical Integration over Space

The action at each time step $S(t_n)$ is computed by integrating the Lagrangian density over the spatial domain:

$$S(t_n) \approx \sum_{i,j} L_{i,j}\, dx\, dy.$$

For improved accuracy, we use the Simpson's rule, a higher-order quadrature method, to perform the integration over x and y:

$$S(t_n) \approx \text{Simpson}\left(\text{Simpson}\left(L_{i,j}, x\right), y\right),$$

where Simpson denotes the application of Simpson's rule over the specified variable.

Handling Numerical Instabilities

During the computation of $L_{i,j}$ and $S(t_n)$, numerical instabilities can arise due to large values of $\phi_{i,j}$ or its derivatives, leading to overflow or NaN (Not a Number) values. To mitigate this, we implement the following precautions:

- **Clipping Field Values:** We restrict the values of $\phi_{i,j}$ to a finite range to prevent overflow:

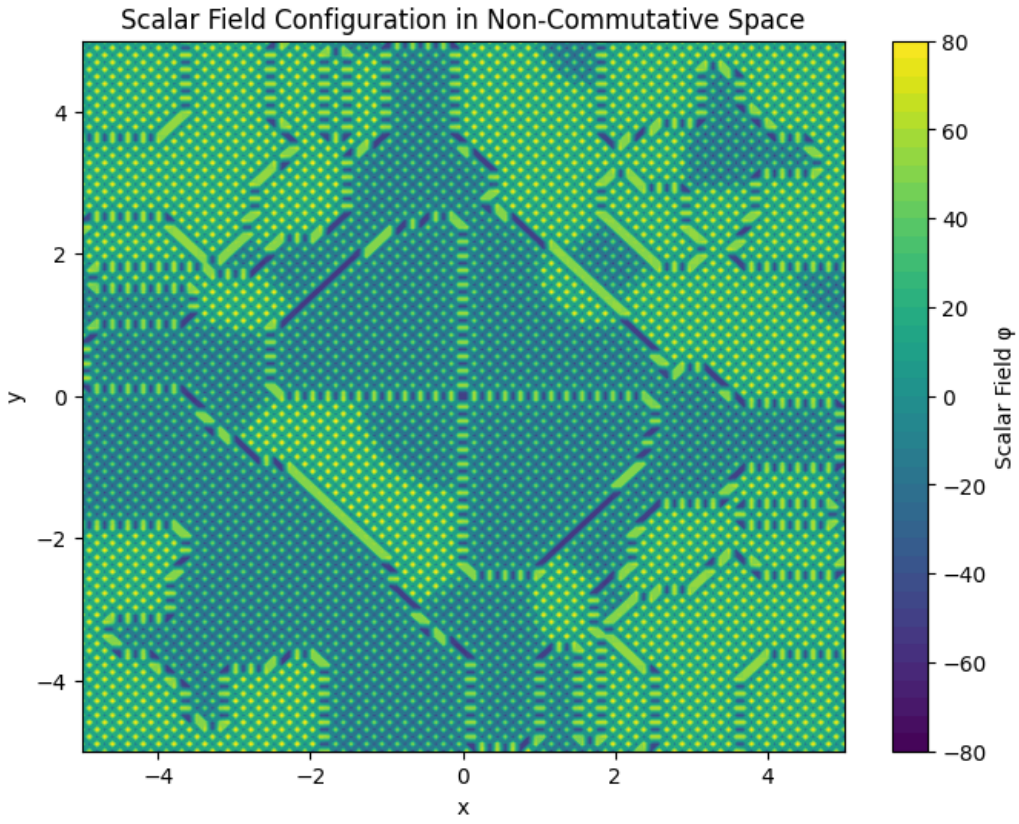
$$\phi_{i,j} = \text{clip}(\phi_{i,j}, -\phi_{\max}, \phi_{\max}),$$

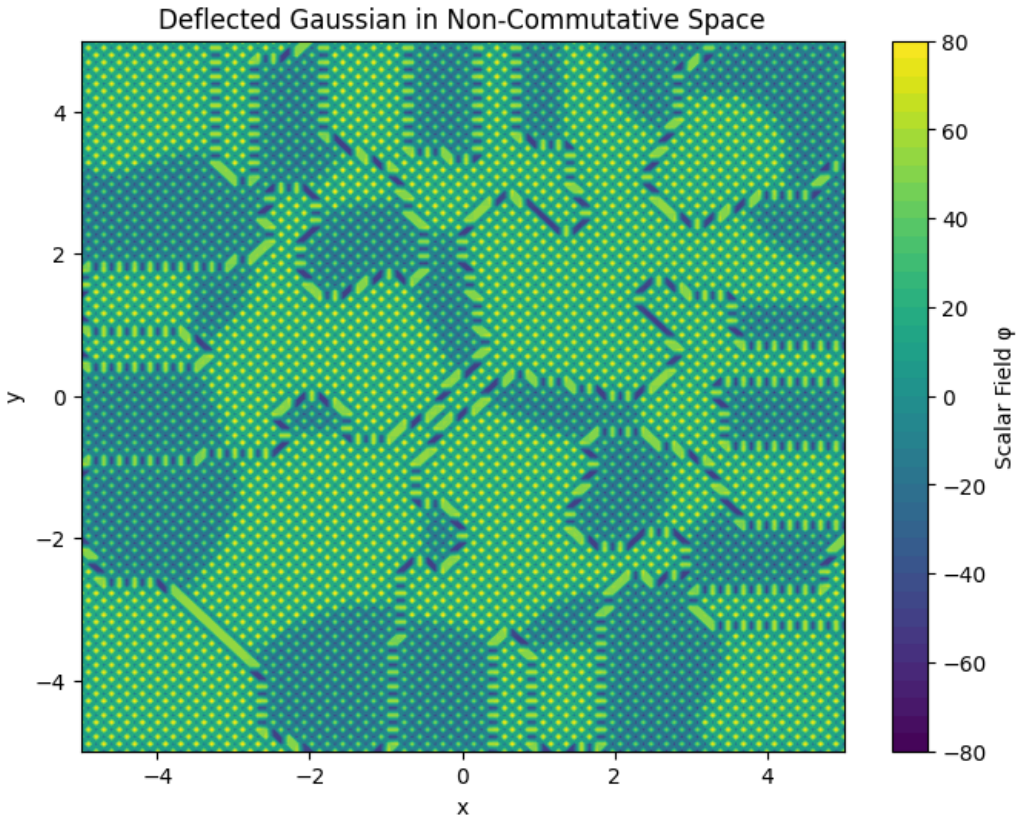
where $\phi_{\max}$ is a predefined maximum value.

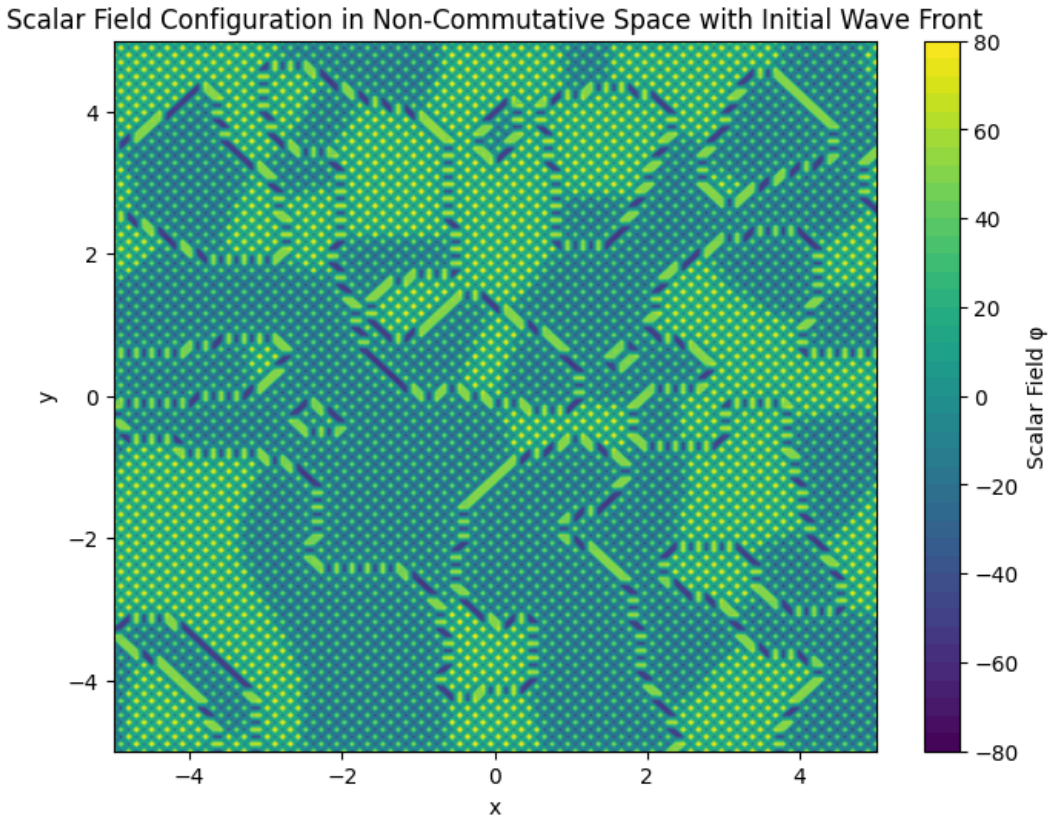
- **Nan and Inf Handling:** We replace NaN and infinite values with finite substitutes:

$$\phi_{i,j} = \begin{cases} 0, & \text{if } \phi_{i,j} \text{ is NaN}, \\ \phi_{\max}, & \text{if } \phi_{i,j} = +\infty, \\ -\phi_{\max}, & \text{if } \phi_{i,j} = -\infty. \end{cases}$$

- **Scaling Initial Conditions and Parameters:** We adjust the magnitude of the initial field configuration and reduce the parameters λ and ϵ to ensure that nonlinear effects do not dominate and cause divergence.


Scalar Field Configuration in Non-Commutative Space
y
x
Scalar Field φ
4
2
0
−2
−4
−4
−2
0
2
4
80
60
40
20
0
−20
−40
−60
−80


Deflected Gaussian in Non-Commutative Space
y
x
Scalar Field φ
4
2
0
−2
−4
80
60
40
20
0
−20
−40
−60
−80



By incorporating these measures, we enhance the robustness of the numerical scheme and obtain meaningful results over the simulation period.

Results and Discussion

Simulation of Scalar Field Configurations

We apply the developed numerical methods to simulate scalar field configurations under various initial conditions, including hyperbolic tangent profiles and Gaussian distributions. The simulations reveal how non-commutative corrections influence the evolution of the field.

The finite difference approximations for mixed derivatives capture the effects of non-commutativity, leading to observable deviations in the field configuration compared to commutative cases. For instance, an initial Gaussian profile experiences deflection due to the mixed derivative term, illustrating the physical implications of spatial non-commutativity.

Analysis of Action over Time

The computed action $S(t)$ provides insights into the dynamical behavior of the system. By tracking $S(t)$ over the simulation period, we can observe trends and identify stable or unstable regimes. The action serves as a diagnostic tool for verifying the consistency of the simulation and the effectiveness of the numerical methods.

Numerical Stability and Performance

The semi-implicit time-stepping scheme demonstrates improved stability compared to explicit methods, allowing for reasonable time steps without sacrificing accuracy. The mixed derivative finite difference scheme effectively approximates the non-commutative corrections, maintaining second-order accuracy.

The computational performance is satisfactory for grids of moderate size (e.g., 100×100). For larger grids or extended simulation times, optimization techniques and parallelization may be considered to enhance efficiency.

Conclusion

We have presented finite difference methods for simulating scalar fields in non-commutative two-dimensional spaces, focusing on the numerical computation of mixed partial derivatives and the action functional. The proposed finite difference scheme for mixed derivatives is accurate and consistent, addressing the challenges posed by non-commutativity in spatial coordinates.

The semi-implicit time-stepping method enhances stability when dealing with stiff nonlinear terms, making it suitable for long-term simulations. By incorporating measures to handle numerical instabilities, we ensure the robustness of the numerical scheme.

The methods discussed are applicable to a variety of problems in theoretical physics where non-commutative geometry plays a role. They provide a foundation for further exploration of non-commutative field theories and contribute to the numerical analysis literature by addressing less documented aspects of finite difference approximations.

References

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